Time in Quantum Mechanics and Related Problems

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Abstract: In this paper we review the role of time in the framework of quantum mechanics. First, we briefly look at the history of the concept of time in physics. Next, the relation between time measurements and the measurement problem is discussed. Then, the time-operator problem is presented, and some proposed solutions are analysed. We also analyse nature of time in the interface between this and future theories. Finally, the various manifestations of time in different interpretations of quantum mechanics are compared.

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1 Introduction

The framework of quantum mechanics is perhaps the most successful to date, providing the most precise, but at the same time strangest, predictions out of all the physical theories. Among the many conceptual and pragmatic difficulties it contains, time is both particularly interesting and problematic.

Our past theories have changed our notion of time more and more, to the point that a technical description of this concept today has notable differences compared to the common understanding. Classical physics contains all our ordinary ideas - and sometimes misconceptions - about time. It has, however, helped us to better understand time's directionality through the concept of entropy. Einstein's theory of relativity, on the other hand, started revolutionising our understanding of time, and this trend has certainly continued in quantum mechanics. Firstly, this is the first physical theory which is not completely deterministic at a fundamental level. This means that one cannot, even in principle, make exact predictions about the future. The outcomes of measurements are not certain, but are supposedly decided - probabilistically and according to specific laws – only at the exact moment of interaction.

In addition to this, quantum mechanics has forced us to consider the effects that the act of measurement itself has on a system. The main mechanism responsible for this influence is that systems collapse when measured, radically altering their evolution. Importantly, this disturbance is closely linked to the amount of information we gain from a measurement. Indeed, since in some cases an absence of detection itself constitutes important information about a system's behaviour, it can collapse the wave-function just as a normal measurement would. For particles travelling freely, for example, attempting to measure their exact time of arrival in a specific region can make them completely reflect off of it, so that no detection is made if the measurement is too strong.

Closely linked to the issue of measuring time distributions is the further problem of time as an operator in the formalism of the theory, which is one that is still being discussed today. The existence of a straightforward universal time operator is simply forbidden theoretically. Nevertheless, concepts such as the time of decay of an unstable atom do not seem ill-defined, and require further exploration before they can be dismissed. Indeed, we are able to obtain theoretical time distributions for various setups, but often only by going beyond the usual simple prescription, which does not appear to be suitable for most problems regarding time measurements.

The principles of locality and causality were once believed to be the main pillars of any physical theory. Quantum mechanics, however, famously exhibits non-local interactions through the phenomenon of entanglement, which have led many to speculate about faster than light communication. Several experimental results have even drawn some to conclude that a quantum system can influence another back in time. More recently, the search for a theory of quantum gravity has stimulated even more discussions on the true nature of causality. When trying to apply quantum theory to the whole universe as a closed system, one is even confronted - depending on the approach - with the problem of having to derive the dynamics we observe from a state in which time does not appear directly. In other words, it seems that time has to emerge from a more fundamental physical condition.

Apart from these important differences, time in the standard interpretation of quantum mechanics is very close to its classical counterpart: it is an external parameter used to describe the evolution of systems. But in the various other interpretations, it can have quite different features. Some hidden-variable theories completely retain the strong determinism of classical physics, while the many worlds interpretation eliminates non-locality and the time-asymmetric notion of collapse. The two-state vector formalism instead attempts to make the time-symmetry of quantum mechanics explicit, by introducing states that evolve

backwards in time. Clearly, even the matter of interpretation has some bearing on the true nature of time.

The purpose of this dissertation is to expand on the conceptual and technical problems just mentioned. In Section 2, we briefly review the main aspects of time in the theories of classical physics, relativity, and quantum mechanics. In Section 3, we discuss various facets of the measurement problem, and how this affects the evolution of systems and our attempts at measuring time. Section 4 is dedicated to the problem of time as an operator, and to some of the approaches that have been formulated to tackle it. Section 5 contains a combination of other problems regarding time in quantum mechanics, including causality and the time problem in quantum gravity. Finally, Section 6 covers some interpretations of quantum physics, and how the role of time differs in each of them.

2 A Brief History of Time in Physics

We can learn a lot about the current consensus on the concept of time, and the approaches we are following to better understand it, if we briefly examine the various roles it has played in our physical theories. From a parameter, to a coordinate in a four-dimensional manifold, to a quantum observable, this quantity has always had a fundamental function in our descriptions of reality.

2.1 Classical Physics

In classical physics, time is a parameter which the other physical quantities can depend on. It is assumed to be universal and external to any system, so a particle does not possess a specific time. Instead, it has a position, velocity etc. at any given time. This notion of time as an external variable has all the properties one would expect from a mathematical realization of what we commonly experience as time: a system possesses all of its physical properties at any moment, and these properties can therefore be regarded as functions of time. Note how this implies that each physical quantity has only one value at any given time, and this gives it its specific role in this theory. It is impossible, for example, for position to have the same role as a universal parameter which the other quantities depend on, as particles can pass through the same point twice, which means time could have two different values at the same position. Time acquires its particular function precisely because the converse is not true: a particle cannot be in two different positions or have two different energies at the same time.

In this theory, knowing the positions and velocities of all the particles in a closed system - which can be the entire universe - would make the future of the system as clear as the present or the past, provided one is able to completely solve all the relevant equations. Interestingly, even though this theory possesses all our common assumptions about time, some of which

have been replaced by later theories, it still paved the way towards our understanding of one of its main features: the so-called arrow of time. The field of statistical mechanics, and with it the concept of entropy, was derived in the framework of classical physics and finally provided a natural law which is clearly asymmetrical in time: the entropy of a closed system increases with time. Today, the concept of entropy is still our best explanation for time's asymmetry, although it has since been expanded and generalised.

2.2 Relativity

The theories of special and general relativity challenged some of our fundamental assumptions about time. The concept of a universal and absolute time was replaced by one that is relative and observer dependent. As a consequence, notions like the universal flow of time and the assignment of a specific absolute time to every event had to be abandoned.

What one has instead is a coordinate that mixes with the three spatial ones to form the unified geometric structure of the spacetime manifold. As such, space and time exhibit the complex behaviours that give us gravity and all the related phenomena. Crucially, despite all this, relativity maintains one of the main features of time, and perhaps of physics itself: the principle of causality. Simultaneity and the rate of the flow of time can be relative, but the notion of an event being able to affect or be affected by another is still something every observer in this theory agrees on. For a more physical notion of time in this theory, as opposed to the mathematical one of a coordinate in a four dimensional manifold, one can consider identical clocks, and how their state of motion and the local structure of spacetime affect their ticking rate. If "time is what a clock measures", then this paradigm reveals the true nature of time in relativity: two clocks moving relative to each other both "see" the other clock ticking more slowly; a clock which is further from a source of gravity ticks faster than one which is closer; different sets of clocks will disagree on what is simultaneous,

depending on their relative state of motion, and so on.

Incidentally, this framework makes this theory's picture of the universe easier to grasp: if one imagines every (massive) particle as carrying its own ideal clock, then each of its interactions will happen at a specific reading of this clock, which every observer will agree on. Therefore, although time measurements can be relative, an underlying universal causal structure is still present. As all the laws of motion in relativity are deterministic, the future evolution of the universe is already completely decided by these laws, by perfect analogy with what was already true in classical mechanics. The description of the universe is thus of a set of particles whose motion is completely prescribed by the theory.

2.3 Quantum Mechanics

What was just discussed is the general vision of the universe given to us by the so-called classical theories of physics. They are to be distinguished from theories that use the framework of quantization. Among the many important differences between these two categories, the treatment of time is certainly an interesting one. At first, this difference may not be obvious, as the parameter of time is similar to that of classical physics: a system is completely described by its quantum state $\hat{\rho}(t)$, which evolves in time - assuming a time independent Hamiltonian - as (here and in the rest of this paper, we take $\hbar = 1$)

$$\hat{\rho}(t) = e^{-i\hat{H}t}\hat{\rho}(0)e^{i\hat{H}t}, \tag{2.1}$$

where $\hat{\rho}(0)$ is the initial state at some reference time, which is taken to be $t = 0$ for convenience. This would be the equivalent of the time evolution of the classical positions and momenta of all the particles in a system (phase space). Using this state, one can then find the probability of measuring some value λ_P of an observable, say corresponding to eigenstate

$\hat{P}$, using

$$p(\lambda_P) = \mathrm{Tr}(\hat{\rho}\hat{P}). \tag{2.2}$$

Here we encounter the first important difference from classical theories regarding the nature of time: the best one can do is calculate probabilities for all the possible values of an observable. Thus, the future evolution of the universe is not completely described by the theory, as after measurement only one of the eigenvalues will be observed, the measured system will collapse to one of the eigenstates and the prescription cannot exactly predict which one it will be.

This loss of classical determinism, however, depends on one's interpretation of quantum mechanics. What was described above is all the standard interpretation has to say about the problem, but others like pilot wave theory or many worlds, manage to retain determinism in some way. This will be examined more in later chapters, but there is another, more operational difference in the treatment of time in quantum mechanics: time as a measurable quantity in this theory was originally, and remains to this day, a controversial and difficult concept.

All the measurable properties of a system were originally thought to be associated with self-adjoint operators. However, as Pauli first realized [45, p.63], there is a problem with the existence of a universal time operator, in that it does not allow for the energy of a system to be bounded from below, in clear disagreement with real physical systems. This theoretical problem of constructing time observables was less apparent in the early days of the theory, since measurements that would tell us, for example, about decay times of individual particles were still impossible to realize [52]. It was, however, already mentioned in several theoretical discussions [9]. Bohr, for example, appreciated the link between time and the measurement problem early on, a link that has only become clearer, as we shall discuss. Today, it is evi-

dent that measurements of quantities such as arrival times, dwell times, decay times etc. are possible, and that trying to find a formalism that successfully predicts these distributions is a valid theoretical enquiry.

As for the arrow of time, this theory is also time reversal invariant and therefore does not include it at a fundamental level. But it has served as a steppingstone to further develop the concepts of entropy and information, which remain of central importance in our descriptions of this concept.

3 The Measurement Problem

As already mentioned, the link between time observables and the measurement problem was realized early on. While discussing the EPR thought experiment, Bohr [16] describes how giving a precise time description in turn requires a more precise account of the interactions between the measuring apparatus and the system. This is by no means unique to time measurements. It is a well-known complication of quantum mechanics that to perform a precise measurement one needs to consider how the devices will affect the object in question. This is one of the major differences between classical and quantum physics, and if one holds the latter to be a complete description of reality, this distinction has to be explainable within the theory itself.

To this end the phenomenon of decoherence, first introduced by Zeh [59], has been the key to describing the quantum to classical transition, and - depending on one's interpretation - is the reason why measurements of time and other quantities do not bring the same complications in classical physics as they do in quantum physics.

3.1 Decoherence

Loosely speaking, decoherence describes the process of entanglement between a system and the environment, and how this suppresses some states of the system which therefore become unmeasurable.

More rigorously (adapted from [51]), take a two-state system s, initially in the superposition: $|\psi\rangle = a|s_1\rangle + b|s_2\rangle$. This can be a particle's spin, or a double-slit setup, or in principle any two-state quantum system that can be in such a superposition before interacting with the environment. Next, we assume the environment E is such that, after interacting with $|s_1\rangle$, it

will a have resulting state $|E_1\rangle$, and similarly $|E_2\rangle$ after interacting with $|s_2\rangle$. In words, we are making the assumption that the environment will end up in different states depending on the state of the system it has interacted with. Moreover, we need the following, more stringent condition: that these two environment states are distinguishable, $|\langle E_2|E_1\rangle| \approx 0$. This assumption, that the environment with which the system in question becomes entangled is sensitive to the different states, is crucial for decoherence. Now, the composite state after interaction will be: $|\Psi\rangle = a|s_1\rangle|E_1\rangle + b|s_2\rangle|E_2\rangle$. Next, consider the reduced density matrix of the system, which contains all the physical information needed to predict any local measurement on it:

$$\hat{\rho}_s = \mathrm{Tr}_E|\Psi\rangle\langle\Psi| = |a|^2|s_1\rangle\langle s_1| + |b|^2|s_2\rangle\langle s_2| + ab^*|s_1\rangle\langle s_2|\langle E_2|E_1\rangle + a^*b|s_2\rangle\langle s_1|\langle E_1|E_2\rangle. \quad (3.1)$$

Note how in any possible measurement on s, the interference terms are given by the "off-diagonal" entries of $\hat{\rho}_s$ (the last two terms in the expression above). Therefore, they will all contain $\langle E_2|E_1\rangle$ or its complex conjugate, which we assumed to be negligible. Thus, no interference will contribute to the statistics, and only the simple states $|s_1\rangle$ and $|s_2\rangle$ will be relevant.

Some important points:

- This process requires work to be undone and is practically irreversible. Information has leaked to the environment, since its two distinguishable states can now be used to infer the state of the system.

- The two states of s were assumed to be the ones that would be "selected" by E. Of course, in a more general model the environment determines these depending on those that correspond to its distinguishable states, but otherwise the key steps of the process

covered above still apply.

- The assumption of distinguishability is not trivially satisfied by large environments. It might look like a property that any classical environment exhibits, but in fact it is not just a statement about the environment itself, but how it interacts with the quantum system. We have many examples of measurements that are carried out in a way which does not destroy all the interference, such as the standard double-slit experiment. But importantly, this is only possible when the environment does not obtain decisive information on the system (i.e. "which-path" information in double-slit). The link between information leak and decoherence is a hallmark of this phenomenon.

- Decoherence is not an instantaneous process. In the example described above, we have a jump from before to after the interaction between system and environment. It is this evolution that takes time, and the details clearly depend on the exact nature of the interaction. As an example, for many setups [46, 41, 34] an accurate model is given by exponential decay:

$$\langle E_i | E_j \rangle \propto e^{-t/\tau_d}, \tag{3.2}$$

where $|E_i\rangle$ and $|E_j\rangle$ are two distinguishable environment states. For a sense of scale the constant τ_d, for decoherence due to collisions, is 0.01 seconds for a large molecule in laboratory vacuum, and 10^{-31} seconds for a dust grain at normal pressure [51].

3.2 Ideal and Weak Measurements

The standard notion of measurement in quantum mechanics, first introduced by von Neumann [43], is that of ideal or projective measurements. The possible outcomes of a measurement are the eigenvalues of a self-adjoint operator, and its orthogonal states are the ones

the system can collapse to, after being measured. This model allows one to exactly predict the subsequent evolution of a system, but its behaviour is necessarily altered in the process.

There are various ways of probing a system which do not exactly determine the state of the system, and consequently do not completely collapse its wave-function. These fall under the category of weak measurements [25], and they can in general be modelled by a Kraus operator $\hat{\mathcal{K}}(q)$, expressed as a function of the measured value q. Kraus operators can actually describe any (completely positive) quantum operation, and for our purposes this includes the interaction with an ambiguous detector. A simple example is given by an operator whose parameter q is distributed as a Gaussian around the eigenvalues of an observable $\hat{O}$:

$$\hat{\mathcal{K}}(q) = \left(\frac{\pi}{\sigma^2}\right)^{1/4} e^{-\frac{1}{2}\frac{(q-\hat{O})^2}{\sigma^2}}. \tag{3.3}$$

As $\sigma \to 0$ this approaches an ideal measurement, but the distribution of q approaches that of the eigenvalues of $\hat{O}$ and consequently interaction with the device causes collapse. Conversely, as $\sigma \to \infty$ this approaches a weak measurement, where the collapse is vanishingly small but the distribution of q is such that the information gained is also negligible. Weak measurements therefore seem to confirm the general principle that the less ambiguous a measurement of a given observable is, the more of a disturbance it will cause on the evolution of a system.

3.3 The Quantum Zeno Effect

The limit of continuous measurements provides an extreme example of how measurement affects the evolution of quantum objects. In the original paper describing this behaviour [23], the lifetime of a decaying particle is examined. The theoretical lifetime, corresponding to unmeasured evolution, is compared to the one which takes measurements into account.

One can show that the lifetime of the particle being monitored becomes arbitrarily large as the frequency of measurements increases. In other words, an unstable particle becomes effectively stable in the limit of continuous measurements. This is an example of the so-called quantum Zeno effect, which in general is the suppression of shifts between quantum states as a result of collapse due to measurement. It has since been extended to many other setups and observed in several different settings [48, 40, 49].

Incidentally, the quantum Zeno effect is ubiquitous when trying to model time observables such as the time of arrival or the dwell time of a particle. Consider, for example, the problem of determining the arrival time in the region $x > 0$, of a particle coming from the region $x < 0$. For this purpose, one can consider a detector monitoring the presence of the particle in the region $x > 0$ at equally spaced time intervals Δt. In a similar way to what was discussed above for the decaying particle, it can be shown [27] that when this kind of pulsed measurement is performed too frequently a particle's behaviour changes drastically: the particle remains in the $x < 0$ region and no detection is made.

Despite this complication, which clearly forbids us from obtaining an ideal time of arrival distribution in this simple way, there are various workarounds which can provide different distributions depending on the type of measurement one wants to employ. A conceptually simple fix [33] is to make the interval between pulsed measurements Δt large enough that the particle can enter $x > 0$. Of course, the limit of very large Δt is just as problematic: if a detection happens at time t, we can only establish that the particle arrived in $x > 0$ at some time between $t - \Delta t$ and t, which is not very useful when Δt is too large. The solution is therefore to also make sure that Δt is small enough that a sufficiently small resolution of the measurement can be obtained. Within this range of possible measurement intervals, one has to check that the obtained distribution is independent of the chosen Δt. This is to make sure that the results are not particularly sensitive to the experimental setup and that one is

actually measuring a meaningful arrival time distribution of the particle.

3.4 Tunnelling and the Larmor Clock

Particles travelling across a rectangular potential barrier are an instructive example when it comes to time measurements, as many of the conceptual pitfalls and difficulties it contains generalise to several other systems. We immediately run into the first difficulty when trying to analyse the time of flight for such particles, since they interact with the barrier in different ways depending on their energy. Even if we limit our observations to the particles that arrive on the other side, which eliminates the ones that reflect off the barrier, we are still left with the problem of tunnelling: if the energy of the particles is less than the potential, an evanescent part of the waveform appears on the far side. When compared to a free particle, this small tunnelling part of the wave-function will lie ahead, by approximately the width of the barrier [54], as if the latter was traversed instantaneously. So, naively, we might expect a superluminal tunnelling time.

For an actual estimate of the time of traversal, we consider the Larmor clock, which has been widely used in the context of defining and measuring quantities like the tunnelling time, the time of traversal and the dwell time of a particle. The idea, first introduced in [12] and [50], is to use the precession of spin due to a uniform magnetic field to infer the time a particle has spent inside it. The magnetic field should be confined to the extent of the potential barrier, and small enough so as not to perturb the particle (which makes this a weak measurement). The component of the particle's spin perpendicular to the field is then predicted to rotate with constant angular speed ω while inside it. Thus, if the final perpendicular spin component is measured to make an angle ϕ compared to the initial one, we calculate the time of traversal to be $\tau_{_y} = \phi/\omega$.

This would be the final answer if we were studying the motion of a classical particle. However, tunnelling is a strictly quantum phenomenon, and this fact brings a couple of complications. First, there is no classical limit to check our observables against, as a classical particle is just unable to cross the potential barrier when it does not have enough energy. Even if we try to apply the classical expression for the time of traversal in this situation, it just yields an imaginary answer. The non-existence of a classical limit is quite problematic, especially for what might be considered a classically motivated question, since we are trying to apply the classical concept of "how long the particle spends inside a region" to this exclusively quantum event. Second, and perhaps more importantly, as first pointed out in [20], the Larmor time as defined above is more closely related to the dwell time of the particle as opposed to the time of traversal. In fact, if we take the dwell time to be, as in scattering processes, $\tau_{_D} = N/j$ - where N is the number of particles within the barrier and j the incoming flux -, then the two are exactly the same. The issue with this is that $\tau_{_D}$ includes both reflected and transmitted particles, so it does not help us estimate a time of traversal as it does not isolate the tunnelling particles. In the tunnelling regime, $\tau_{_y}$ - or, equivalently, $\tau_{_D}$ - tends to 0 as the energy E of the incoming particle goes to 0. We might interpret this as tunnelling particles spending less time in the barrier when they have less energy, except of course in this limit the tunnelling amplitude is too small compared to the reflected one. This means that the contribution to $\tau_{_y}$ from the reflected part dominates. So all we are able to discern is that, as expected, reflected particles spend a vanishingly small amount of time inside the barrier.

In [20], the following quantum phenomenon is also pointed out: even if the incoming particles are spin-polarised only in the direction perpendicular to the magnetic field, they of course still effectively interact with it as "50% spin up and 50% spin down". Now, a particle with its spin aligned with the magnetic field has lower energy compared to one with its spin aligned in the opposite direction. In practice, these two facts together mean that the tunnelling

probability for the spin down part of the wave-function is more exponentially suppressed. The upshot is that the spin of the tunnelled particles will effectively be rotated in the direction of the magnetic field. For small magnetic fields, this effect is of linear order in the field, which is the same as the Larmor precession described above. The amount of deflection from this rotation of the spin can therefore also be used to obtain another candidate τ_z for the time of traversal. This particular expression has a finite value as E approaches 0, which solves the tunnelling limit problem. This time though, we have the opposite problem, as τ_z becomes vanishingly small in the limit of high E. Therefore, to obtain a more universal value which takes into account the full deflection of the spin, and which also agrees with a separate estimate [21] of the tunnelling time, in [20] the following expression for the total time of traversal is suggested:

$$\tau_T = \sqrt{\tau_y^2 + \tau_z^2}. \tag{3.4}$$

Although no consensus has been reached on the proper definition of the tunnelling time - especially since it is difficult to find quantities that generalise well and are defined for all setups -, we are still able to provide some tentative answers such as this one, often depending on the specific measurement scheme we have chosen. We will return to the tunnelling time problem when discussing the path integral formulation in the next chapter.

Finally, as regards the concerns of faster than light travel in tunnelling processes, it has been argued in [22] among many others, that none of these effects can be used to send information faster than what is allowed by the principle of relativity. This is in the same vein of similar seemingly superluminal effects in electrodynamics, which have also been reconciled with causality [17].

4 Time Observables

The fact that the existence of a universal time operator would be problematic was realized early on. The nail in the coffin, which is often mentioned when discussing time operators, was Pauli's theorem, which goes as follows: assume such a self-adjoint operator $\hat{T}$ exists. Its canonical commutation relation (CCR) with the Hamiltonian, $[\hat{T}, \hat{H}] = i\hat{I}$ (where $\hat{I}$ is the identity), means that the time operator generates translations in energy in a way that is completely analogous to position and momentum. That is, if $|E\rangle$ is an eigenstate of $\hat{H}$ with eigenvalue E, then we must have

$$\hat{H}e^{-i\mathcal{E}\hat{T}}|E\rangle = (E + \mathcal{E})|E\rangle, \tag{4.1}$$

for any real number $\mathcal{E}$, so that we have a new eigenstate corresponding to the eigenvalue $E + \mathcal{E}$. This in turn means that the spectrum of $\hat{H}$ spans the entire real line, and in particular it cannot be discrete or bounded from below. In the early days of the theory, when it had been postulated that every observable must correspond to a self-adjoint operator, the preceding argument seemed to completely forbid the existence of an operator that would yield predictions for time measurements in all systems of physical interest.

There are, however, some mathematically subtle assumptions which go into Pauli's theorem:

Firstly, we now know that observables are not necessarily linked to self-adjoint operators. The more general positive operator-valued measures (POVMs) are not necessarily self-adjoint and can be linked to observables.

Furthermore, and perhaps more subtly, a time operator that is conjugate to the Hamiltonian does not necessarily need to be non-self-adjoint, as Pauli's theorem seems to imply. The reason is that the extension of the spectrum of $\hat{H}$ to the entire real line is only valid if the domain of the CCR has some specific properties, which although common in simple

examples, are by no means universal. Therefore, one can still have a self-adjoint time operator conjugate to a semi-bounded and/or discrete Hamiltonian, as long as the corresponding CCR is valid only in a "non-standard" domain.

4.1 Time Observables as POVMs

One can arrive at the notion of POVMs when trying to find a general operator associated to a given observable. If the range of possible values the observable can take is ν, then for subsets $s_i \subset \nu$, the probabilities p_{s_i} that the outcome of measuring the observable is in s_i should satisfy some standard axioms like additivity and positivity. Given these, one can derive [8] that there should exist a set of positive operators $\hat{A}_{s_i}$, each associated with one of the subsets s_i, which yield the corresponding probabilities as

$$p_{s_i} = \mathrm{Tr}(\hat{A}_{s_i}\hat{\rho}). \tag{4.2}$$

The $\hat{A}_{s_i}$ need to satisfy various constraints, such as $\hat{A}_{s_i \cap s_j} = \hat{A}_{s_i} + \hat{A}_{s_j}$ for disjoint s_i and s_j, and so on, all in accordance with the axioms of probability. A POVM is precisely this set of operators, each associated to a set of real numbers and all satisfying the required constraints. It is clear how the standard projective measurements belong to this more general class of POVMs, where each operator is the projector onto one of the eigenstates and its associated set of real numbers contains only the corresponding eigenvalue. In this case, the operators are also all orthogonal, a feature which POVMs do not necessarily exhibit.

As an example of a specific time operator realized as a POVM, following [8], consider the classical time of arrival at the origin t for a particle at x with momentum p,

$$t = -\frac{mx}{p}. \tag{4.3}$$

One of the reasons why finding well-defined time operators is not an easy task is that the corresponding classical expressions, like the one above, often include non-commuting operators, and their quantization is therefore not unique. A common approach is to apply the symmetrization rule. Doing this for the quantity above, we obtain the so-called Aharonov-Bohm operator, a time of arrival operator for the free particle:

$$\hat{T}_A = -\frac{m}{2}(\hat{x}\hat{p}^{-1} + \hat{p}^{-1}\hat{x}). \tag{4.4}$$

By Pauli's theorem, we know that this operator is either not conjugate to the Hamiltonian, or it is not self-adjoint. Using $[\hat{x}, \hat{p}] = i\hat{I}$, it is easy to show that in fact $[\hat{H}, \hat{T}_A] = i\hat{I}$, where $\hat{H}$ is the free Hamiltonian $\frac{1}{2m}\hat{p}^2$. Therefore, we infer that $\hat{T}_A$ is not self-adjoint. However, it might still be used to define a valid POVM. To that end, let us move to the energy representation, where the operator takes the suggestive form $-i\frac{\partial}{\partial E}$. There are a few complications with the domain of the operator and the fact that the degeneracy in energy means we must have two sets of generalized eigenfunctions, one for positive and one for negative momenta. In the end, one finds that the probability density corresponding to this operator is the following:

$$\Pi(t) = \left| \int_0^\infty dE \frac{e^{-iEt}}{\sqrt{2\pi}} \left(\frac{m}{2E}\right)^{\frac{1}{4}} \Psi(+\sqrt{2mE}) \right|^2 + \left| \int_0^\infty dE \frac{e^{-iEt}}{\sqrt{2\pi}} \left(\frac{m}{2E}\right)^{\frac{1}{4}} \Psi(-\sqrt{2mE}) \right|^2 \tag{4.5}$$

where $\Psi(p) = \langle p|\Psi\rangle$ is the state in the momentum representation. Notice the clear split between positive and negative momenta, corresponding to the first and second term respectively. We have thus arrived at the POVM associated with the $\hat{T}_A$ operator.

$\Pi(t)$ happens to be the ubiquitous Kijowski distribution, which he first derived [38] by considering a specific set of functionals associated with time of arrival, and then demanding minimum variance. This is the ideal distribution of arrival times for a free particle, but as such it is difficult to measure in practice. Indeed, this general difficulty of quantum mechanics, that measurement methods do not readily follow from operators, is still very much a

problem for POVMs [8]. Several theoretical measurement models which have been devised seem to confirm this distribution for large momenta, but they differ for small momenta, as reflection off the detector becomes important in this regime [31].

It is possible that this derivation based on $\hat{T}_A$ can be adapted to some interacting cases, or that Kijowski's method based on optimization is transferable to other systems. Regardless, this example, though not readily generalizable, shows that it is possible to obtain a valid and well-defined distribution for a time observable using POVMs.

4.2 The Path Integral Approach

Another time observable of physical interest is the dwell time, which is the fraction of a time interval, say between t_1 and t_2, that a particle spends inside a specified region in space Ω. Feynman's path integral formulation of quantum mechanics [26] is well-suited for this problem, as one can selectively sum over only the paths that spend a specific amount of time τ in Ω, thus obtaining a distribution in dwell time τ.

In what follows $S[x(t)]$ is the classical action as a functional of the classical path $x(t)$ of the particle:

$$S[x(t)] = \int_{t_1}^{t_2} dt \left(\frac{1}{2} m \left(\frac{dx}{dt} \right)^2 - V(x) \right) \tag{4.6}$$

Taking the initial time as $t_1 = 0$ and evolving the state up to time T, the solution $\Psi(x,t)$ to the 1D Schrödinger equation can be written via the path integral as

$$\Psi(x,T) = \int dx' \int_{x'(0)}^{x(T)} Dx(t) e^{iS[x(t)]} \Psi(x',0), \tag{4.7}$$

where the second integral is over paths which start at x' for $t = 0$ and end at x for $t = T$. We now look for a restriction to this integral that allows us to select among these only the

paths that spend a time τ inside the region Ω. For this, we need the classical dwell time τ_c, which is obtained by summing over only the time that the particle spends in Ω:

$$\chi_\Omega(x) = \begin{cases} 1 & \text{if } x \text{ is in } \Omega \\ 0 & \text{otherwise} \end{cases} ; \tag{4.8}$$

$$\tau_c[x(t)] = \int_{t_1}^{t_2} dt \chi_\Omega(x(t)). \tag{4.9}$$

With this, we simply insert a $\delta(\tau_c[x(t)] - \tau)$ in the path integral, to only select for paths which spend an amount of time τ in Ω:

$$\Phi(x, T|\tau) = \int dx' \int_{x'(0)}^{x(T)} Dx(t) \delta(\tau_c[x(t)] - \tau) e^{iS[x(t)]} \Psi(x', 0). \tag{4.10}$$

Interestingly, Φ can be written [54] in terms of a different wave-function $\Psi_W(x,t)$, which still solves the Schrödinger equation but with the modified potential $V(x) + W\chi_\Omega(x)$, in the following way:

$$\Phi(x, T|\tau) = \frac{1}{2\pi} \int dW e^{iW\tau} \Psi_W(x, T). \tag{4.11}$$

Note this has the suggestive form of a Fourier transform in the variables W and τ. In any case, to simulate measurement we coarse grain this distribution by convolution with a function $G(\tau)$, which ideally is sharply peaked with some width $\Delta\tau$ around $\tau = 0$:

$$\Psi(x, T|\tau) = \int_{-\infty}^{\infty} d\tau' G(\tau - \tau') \Phi(x, T|\tau'). \tag{4.12}$$

Finally, this is the amplitude that yields the probability of the dwell time to be within $\frac{\Delta\tau}{2}$ of τ, for a particle in x at time T, which we may write as:

$$p(x, T|\tau) = |\Psi(x, T|\tau)|^2. \tag{4.13}$$

Of course, using this one can find other useful quantities, such as the average dwell time for particles that arrive at a specific position x at time T, or the more general average dwell time:

$$\langle \tau \rangle = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tau dx \ \tau \cdot p(x,T|\tau)}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tau dx \ p(x,T|\tau)}. \tag{4.14}$$

Interestingly, as shown in [55], this evaluates to the complex quantity $\langle \tau \rangle = \tau_y + i\tau_z$ for the rectangular barrier discussed in Sect. 2.4, and it thus accounts for both the precession and rotation of the spin in the Larmor clock setup, with its magnitude being exactly the tunnelling time of (3.4). As an aside, note the average value of what should be a real quantity comes out complex. For more on this see [56].

Despite all this, the definition of the tunnelling time remains problematic. The fundamental reason, as illustrated in [54], seems to lie in interference and collapse. To understand this, note how the standard wave-function has been continuously "unfolded" into subamplitudes in (4.10) via the variable τ. Indeed, we may recover the regular amplitude by just integrating the unfolded one over all possible dwell times:

$$\Psi(x,T) = \int_0^T d\tau \Phi(x,T|\tau). \tag{4.15}$$

What this unfolding suggests is that the various paths with different dwell times interfere with each other, and that measuring τ disrupts these interactions, by analogy with a "which-path" experiment. Indeed, a measurement ideally discards all the paths with τ outside a certain range, and it involves the interference of the remaining ones. In this sense, within standard quantum theory, the dwell time of an unmeasured particle is as poorly defined as its position or the path it has taken. These quantities only take on specific values when they are measured, and this in turn means that the evolution of the particle must be inevitably

perturbed. The details, as always, depend on the type and frequency of the measurements. At any rate, the above is an example of how one can obtain time distributions without having to resort to the standard prescription of operators and eigenfunctions, along with their associated difficulties.

4.3 The Time-Energy Uncertainty Principle and CCR

As equation (4.11) in the last section might suggest, measuring the dwell time might perturb the potential in Ω, so that τ and W satisfy an uncertainty relation as any pair of conjugate variables would. Of course, this requires a particular detector model to be constructed, but indeed we have [54]:

$$\Delta\tau\Delta W \geq 1 \qquad \text{(with } \hbar = 1\text{)}, \tag{4.16}$$

so that measuring the dwell time more accurately means having more uncertainty in the potential in the region of interest.

The uncertainty relation between time and energy was originally thought to be universal, as their operators were believed to be conjugate in the general case [37]. However, proofs of this relation were scarcely provided. Modern proofs turn out not to be so general, and almost all of them are far from the standard one regarding conjugate observables. A widely applicable one (from [29]) goes as follows. For a time-independent operator $\hat{O}$, we have the usual uncertainty relation with the Hamiltonian:

$$(\Delta E)^2(\Delta O)^2 \geq \left(\frac{1}{2i}\langle[\hat{H},\hat{O}]\rangle\right)^2. \tag{4.17}$$

Using in the Schrödinger equation, it can further be shown that for such an operator

$$\frac{d\langle\hat{O}\rangle}{dt} = i\langle[\hat{H}, \hat{O}]\rangle. \tag{4.18}$$

Plugging this into (4.17) and taking the square root yields

$$\Delta E \Delta O \geq \frac{1}{2}\left|\frac{d\langle\hat{O}\rangle}{dt}\right|. \tag{4.19}$$

Finally, to get the form of a standard uncertainty relation, we define the "time uncertainty" as

$$\Delta T = \frac{\Delta O}{\left|d\langle\hat{O}\rangle/dt\right|}, \tag{4.20}$$

so that (4.19) turns into our final expression:

$$\Delta E \Delta T \geq \frac{1}{2}. \tag{4.21}$$

One can choose any operator $\hat{O}$, and this relationship tells us that if its expectation value changes too quickly, the uncertainty in the energy must be large. Conversely, if the energy uncertainty is very small, the expectation value of any observable must change slowly. However, although ΔE represents the standard uncertainty in energy, the "time uncertainty" above is completely different. The price we pay for complete generality is that the ΔT we have defined does not directly involve any specific time operator. Indeed, one could in principle have different kinds of measurements of times or energies in which a theoretical time-energy uncertainty relation of this kind is not satisfied [19].

As an example of this, consider the following model due to Aharonov and Bohm [1]: a

two-particle system in 1 dimension with the following Hamiltonian

$$\hat{H} = \frac{1}{2M_x}\hat{P}_x^2 + \frac{1}{2M_y}\hat{P}_y^2 + \hat{Y}\hat{P}_y g(t), \tag{4.22}$$

where g(t) is constant for a short period of time, say Δt, and zero otherwise. The Heisenberg operators for the momenta after the interaction are as follows

$$\hat{P}_x(t) = \hat{P}_x(0), \qquad \hat{P}_y(t) = \hat{P}_y(0) - \hat{P}_x(0) g_0 \Delta t, \tag{4.23}$$

where g_0 is the constant value of $g(t)$ during the interaction. Given this, if we wish to measure P_x with some given accuracy ΔP_x using the "probe" Y, we should measure the values of P_y before and after the interaction, say at times 0 and t, and then compare the two. To achieve the specified accuracy, the probe must be sensitive enough to changes in P_x. In particular, if P_x changes by δP_x, we require that the corresponding probe change $\Delta\big(P_y(t) - P_y(0)\big)$ be greater than the uncertainty of the probe's initial state $\Delta P_y(0)$. Now from the Heisenberg equation for $\hat{P}_y(t)$ above, the probe change is given by:

$$\Delta\big(P_y(t) - P_y(0)\big) = \Delta P_x g_0 \Delta t, \tag{4.24}$$

so that our accuracy condition is

$$\Delta P_x g_0 \Delta t > \Delta P_y(0). \tag{4.25}$$

By having a strong enough interaction constant g_0, one can clearly make both ΔP_x and Δt arbitrarily small, meaning the uncertainty in energy can be made small while also keeping the interaction time short, in clear disagreement with a supposed time-energy uncertainty relation.

More generally, there seems to be no limit to the accuracy of time measurements if these are carried out externally. On the other hand, internal measurements from within the system satisfy uncertainty relations such as (4.21) [5].

Of course, for time operators which satisfy the CCR with the Hamiltonian, we can recover (4.21) from the usual uncertainty relation, with ΔT now representing the uncertainty of the corresponding time observable. As covered before, Pauli's theorem represents the main obstacle to the existence of such a time operator. Another mathematical subtlety concerning its formulation, other than the existence of POVMs, is the domain of validity of the CCR between the Hamiltonian and a possible time operator.

The assumption behind the theorem is that this domain is such that it forces the spectrum of the Hamiltonian to be unbounded below. In particular, this actually requires the CCR to be valid in a dense subspace of the Hilbert space, and this subspace to be invariant under the action of either operator. An important property of the CCR [27] is that no two operators can satisfy it in the whole Hilbert space, but only in a proper subspace. While it is true that for the most common examples, like the position-momentum pair, this subspace is often both dense and invariant, this assumption does not have to hold in general, as it has to do, among other things, with the topology of the Hilbert space under consideration. In particular, it turns out one can have operators that exhibit the properties one might require of a time operator, but they only satisfy the CCR in a non-dense subspace [27]. Therefore, we may have to relax these constraints in specific cases, as restrictions on a time operator that might seem physical and valid in one system - for a specific Hilbert space and with a specific Hamiltonian - do not necessarily apply to or have solutions in all systems.

4.4 The Dwell Time Operator

As a final example of a time operator, consider the dwell time of a particle (from [10]), this time within the standard quantum measurement formalism. The dwell time operator can be written as:

$$\hat{T}_\Omega = \int_{-\infty}^{\infty} dt \hat{\chi}_\Omega(t), \tag{4.26}$$

where $\hat{\chi}_\Omega(t)$ can be seen as the Heisenberg operator version of (4.8), with:

$$\hat{\chi}_\Omega(0) = \int_\Omega dx |x\rangle\langle x|, \tag{4.27}$$

as it projects states onto the region Ω. The fact that allows us to once again side-step Pauli's theorem is that this particular operator actually commutes with the Hamiltonian:

$$\hat{T}_\Omega e^{-i\hat{H}t} = \int_{-\infty}^{\infty} d\tau e^{i\hat{H}\tau} \chi_\Omega(0) e^{-i\hat{H}(\tau+t)} = \int_{-\infty}^{\infty} d\tau e^{i\hat{H}(\tau-t)} \chi_\Omega(0) e^{-i\hat{H}\tau} = e^{-i\hat{H}t} \hat{T}_\Omega. \tag{4.28}$$

Physically, this is because $\hat{T}_\Omega$ represents an interval of time rather than an instant of time. Note, however, the need to use these particular limits of integration in the proof above, so that no "restricted" version of this operator necessarily commutes with the Hamiltonian. The fact that these two operators commute allows us to simultaneously diagonalize them. For Hamiltonians with plane waves $|\pm k\rangle$ as eigenfunctions, with continuous spectrum $E = k^2/2m$, it turns out that we have two eigenvalues for $\hat{T}_\Omega$ at any given energy. We denote the two corresponding eigenvectors as $|t_\pm(k)\rangle$, with $k > 0$, so that a resolution of the identity is given by:

$$\hat{I} = \int_0^{\infty} dk \Big(|t_+(k)\rangle\langle t_+(k)| + |t_-(k)\rangle\langle t_-(k)| \Big). \tag{4.29}$$

We can now compute the generating function of the dwell time distribution for a general state $|\Psi\rangle$ as follows:

$$F(\omega) = \langle\Psi|e^{i\omega\hat{T}_\Omega}|\Psi\rangle = \int_0^\infty dk\Big(e^{i\omega t_+(k)}\langle\Psi|t_+(k)\rangle\langle t_+(k)|\Psi\rangle + e^{i\omega t_-(k)}\langle\Psi|t_-(k)\rangle\langle t_-(k)|\Psi\rangle\Big), \tag{4.30}$$

where we have inserted the identity and then plugged in (4.29). Finally, the corresponding dwell time distribution is:

$$T_\Omega(t) = \int_{-\infty}^\infty \frac{d\omega}{2\pi}e^{-i\omega t}F(\omega) = \int_0^\infty dk\Big(\delta\big(t - t_+(k)\big)\big|\langle\Psi|t_+(k)\rangle\big|^2 + \delta\big(t - t_-(k)\big)\big|\langle\Psi|t_-(k)\rangle\big|^2\Big). \tag{4.31}$$

The interesting features of this dwell-time distribution, which distinguish it from a classical one, are generally caused by the presence of the two eigenvalues $t_\pm(k)$. Note they do not always correspond to the two possible signs of momentum for a given energy. The presence of two eigenvalues causes the distribution to become bimodal, even in cases where one might expect a unique dwell time, such as when the momentum wave-function is highly localized.

5 Other Problems Related to Time

Time has many functions in quantum mechanics just as in most physical theories. Its precise role and fundamental properties can perhaps be unclear in our current theories, where we always seem to have to resort to some notion of an external parameter, measured by ideal clocks which have nothing to do with the system under consideration. But there is much to learn about its various features form many frontiers of research. The following topics are not all directly related to time, but it naturally appears in their investigation, and they all have important consequences for some of its basic properties.

5.1 Measuring External Time: Clocks

So far, we have discussed issues with measuring what one might call observable time (time of arrival, decay time, dwell time etc.). External time, on the other hand, is the notion we have more experience with and is more relevant in most experiments. It is what the external clock measures, which one sets as a reference for the various interactions and measurements that will take place.

Such an instrument typically has two main components: a periodic system, whose period can be predicted theoretically, and a probe, which can count the number of times the main system has undergone its "oscillation" and display it in some way. The notion of a probe is perhaps less relevant for ancient devices like hourglasses, where we ourselves act as a probe - although one could argue that the light bouncing off of the sand and through the glass is really what is "probing the system" -, and there is no need to consider our influence on the system. But even in a classical pendulum clock, we have to account for the influence that the "probe" gears might have on the oscillating pendulum they are connected to. More modern examples present even more interactions that we have to take into account, especially as the

most accurate ones today all involve some type of quantum system, which further complicates things in light of the measurement problem discussed above.

An example of a modern atomic clock [58] roughly works by preselecting the specific energy level of atoms of a specific element, for example by interaction with a magnetic field, and then checking that they interact with the uniform bath of electromagnetic radiation that is being monitored. It is a well-known prediction of quantum theory that an atom will only change energy level by interacting with an incoming photon only if its frequency is the one needed to make the transition. In this way, skipping a few details, the clock continuously checks that the electromagnetic frequency is approximately the right one and then, based on the number of cycles of this radiation, counts the time. The modern definition of the unit of time is precisely based on such a system: the atoms in question are caesium-133 atoms in their ground state, and the radiation that makes them transition to a specific energy level undergoes - by definition of the second - exactly $9,192,631,770$ periods of oscillation per second.

Incidentally, a well-known theorem from Unruh and Wald [57] shows that it is impossible for a system to serve as a perfect quantum clock that measures Schrödinger time, that is, the variable that appears in the Schrödinger equation. To prove this, we only have to restrict the Hamiltonian of this clock to be bounded below, and require the quantum operator $\hat{T}$ - which provides the time observable - to have certain properties that define a perfect clock.
1) We require the existence of states $|1\rangle$, $|2\rangle$, $|3\rangle, ...$ where $|n\rangle$ is an eigenstate of the projection operator onto the interval of the spectrum of $\hat{T}$ centered around some eigenvalue t_n, with these eigenvalues ordered ($t_1 < t_2 < t_3 < ...$). Note we have not assumed that the spectrum of $\hat{T}$ is continuous, nor dismissed this possibility.
2) Next, for the clock to run forward in time, there has to be a non-zero probability for any state $|n\rangle$ to evolve into a state $|m\rangle$ with $n < m$, at least for one such m.
3) Finally, for the clock to *only* run forward in time, we require the amplitude for $|n\rangle$ to

evolve into $|m\rangle$ to vanish for all m with $n > m$.

We now define and study the following function:

$$f_{mn}(t) := \langle t_m | e^{-i\hat{H}t} | t_n \rangle. \tag{5.1}$$

As a function of the complex variable t, $f_{mn}(t)$ is holomorphic in the lower half of the complex plane since by assumption $\hat{H}$ is bounded below. Now, requirement (3) means that $f_{mn}(t) = 0$ for all real $t > 0$ when $n > m$. Because of the holomorphy of this function, we can conclude that $f_{mn}(t) = 0$ on the entire real line when $n > m$. Now, because the evolution is unitary $f_{mn}(t)$ has the following property:

$$f_{mn}(t) = \langle t_m | e^{-i\hat{H}t} | t_n \rangle = \langle t_n | e^{+i\hat{H}t} | t_m \rangle^* = f^*_{mn}(-t). \tag{5.2}$$

This means that, for real t, we can write the function for $n < m$ in terms of the function for $n > m$. Therefore, since the latter has been shown to vanish for all real t, the former also has to vanish, in contradiction with property (2). Notice this is very distinct from the aforementioned Pauli's theorem, as by itself it does not negate the existence of a time operator, only of one which satisfies all the properties (1)-(3) above. We conclude that for such a clock to exist, it must have at least some non-zero probability of running backwards.

5.2 Space and Time

In standard quantum mechanics, space and time stand on unequal footing. Indeed, almost all the problems discussed so far do not apply to position observables and operators. In 1 dimension, there is a well-defined position operator $\hat{x}$ valid for all systems, which is self-adjoint and has a complete spectrum of orthogonal eigenstates.

To examine this difference further, consider the phenomenon of entanglement. The clas-

sic example of two entangled particles is the singlet state of two spin-$\frac{1}{2}$ particles

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle). \tag{5.3}$$

The two particles are said to be entangled as one can gain information about both particles at the same time by just measuring the state of one of them. In terms of standard measurement theory, one can measure the state of one particle to be spin down - causing collapse -, and the state of the other particle will also instantly collapse to spin up. We can also have a notion of "temporal entanglement", although the mapping is not a straightforward one. Instead of two distinct systems which have interacted in some way in the past, we explore the correlations between two states of the same system at two distinct moments, with time evolution in between. The evolution can be unitary but more generally is completely positive, so that the system could interact with an environment, ideal or weak measurements could be performed on it, or it could evolve unperturbed.

These two types of entanglement, spatial and temporal, share some similarities but also have fundamentally different properties. First, when considering only a pair of entangled states, the main difference is that operators corresponding to measurements performed on two different systems commute. In the temporal case, the equivalent situation would be performing two measurements on the same system at two different times. But in this case the operators clearly do not commute in general, so the correlations must be of a fundamentally different character. If we only consider weak measurements, however, there exists a map between weakly measured spatially and temporally entangled states, in other words between a bipartite system and a system evolving in time [42]. Within this map, a maximally entangled bipartite system corresponds to unitary evolution in time. This is not surprising, as for an evolving system which does not interact with an environment, the two measurements are in a sense maximally entangled, as one can exactly predict the evolution between them. On

the other extreme, a bipartite system which is not entangled - or equivalently, which can be written as a pure product of states - corresponds to a system which is pre-selected to some state and then post-selected to another state.

Another difference between spatial and temporal correlations is that spatial entanglement is "monogamous". That is to say, if two systems are maximally entangled, neither of them can be entangled with a third system. In the temporal setting on the other hand, if one performs weak measurements at three different times, the three possible pairs of measurements can all be maximally correlated [42]. As one might expect, for strong measurements the situation changes, as for a succession of three measurements in time the second one destroys the correlation between the first and last [18]. Still, the presence of "polygamous" correlations in time represents a clear discrepancy between entanglements in space and time.

Perhaps the most striking feature of entanglement, and one of the strongest departures from classical theories, is its non-locality. As already mentioned, an entangled particle will behave as if it is being measured and collapse when the other is measured, and this collapse occurs instantly, irrespective of their distance. This phenomenon seemingly violates one of the principles of special relativity, that no signal can travel faster than the speed of light. In other words, two points which are space-like separated cannot be causally connected to one another. However, it is a well-known theorem of standard quantum mechanics that no information can possibly be transferred using entanglement. This simple argument goes as follows [47]: two measurements are made in succession on the two sub-systems A and B of a bipartite system $\hat{\rho}$. We model a measurement on sub-system A as

$$\hat{\rho} \to \hat{\rho}' = \sum_m \hat{A}_m \hat{\rho} \hat{A}_m^\dagger, \tag{5.4}$$

where the A_m are Kraus matrices satisfying $\sum_m \hat{A}_m^\dagger \hat{A}_m = \hat{I}$. Complete information on the state of B (immediately) after A has been measured is given by the partial trace:

$$\hat{\rho}_B = \mathrm{Tr}_A \Big(\sum_m \hat{A}_m \hat{\rho} \hat{A}_m^\dagger \Big). \tag{5.5}$$

Now for the crucial step: we assume that the operators $\hat{A}_m$ only act on the sub-system A, and since the trace is also in A, we can use the cyclicity of the trace to bring $\hat{A}_m^\dagger$ to the left inside the trace. Using the defining property of the Kraus operators to carry out the sum, we are then simply left with

$$\hat{\rho}_B = \mathrm{Tr}_A(\hat{\rho}), \tag{5.6}$$

and we can conclude that B cannot discern whether or not A has carried out any measurements, as their statistics are not affected at all.

This is not to say that the measurement in A does not affect the state of B. In fact, from the point of view of the observer at A, the sub-system B could now be in a collapsed state, depending on the measurement and whether the two were entangled. The observer at B can subsequently receive this information, which would change their knowledge of the system, but crucially this requires a causal connection. The only way to obtain the relevant statistics without communicating with A is by use of the partial trace, which as shown above destroys any correlation between the two sub-systems. An important assumption in this theorem is that measurements at A do not directly act on B in any way. This, as far as we know, must be the case if the two systems are space-like separated. For this reason, the principle of relativity is not broken in quantum mechanics and the causal structure of events is still well-defined.

In this case, examining the corresponding problem in the context of "time entanglement" would not be as problematic. Time is, in this sense, quite different from space even in rel-

ativity, as information transfer between time-like separated points is allowed, and physical particles can travel along time-like - or null - as opposed to space-like curves.

5.3 Causality and the Delayed Quantum Eraser Experiment

Causality is of central importance in physics, but hints from our search for a theory of quantum gravity are motivating some to question its fundamental status. As an example, by applying the logic of quantum mechanics to the structure of spacetime [61], one could in principle have superpositions of different mass and energy distributions, which means superpositions of spacetime geometries, and therefore a possible superposition of temporal orders of events. One could argue that decoherence would suppress such mass superpositions, as they produce very distinguishable environmental states. Crucially though, this is not the same as eliminating them entirely. Thus, in such a universe a notion of universal causality becomes invalid, and this concept only holds true given the right limits, by analogy with the emergence of classical physics from standard quantum mechanics, or of the latter from quantum field theory.

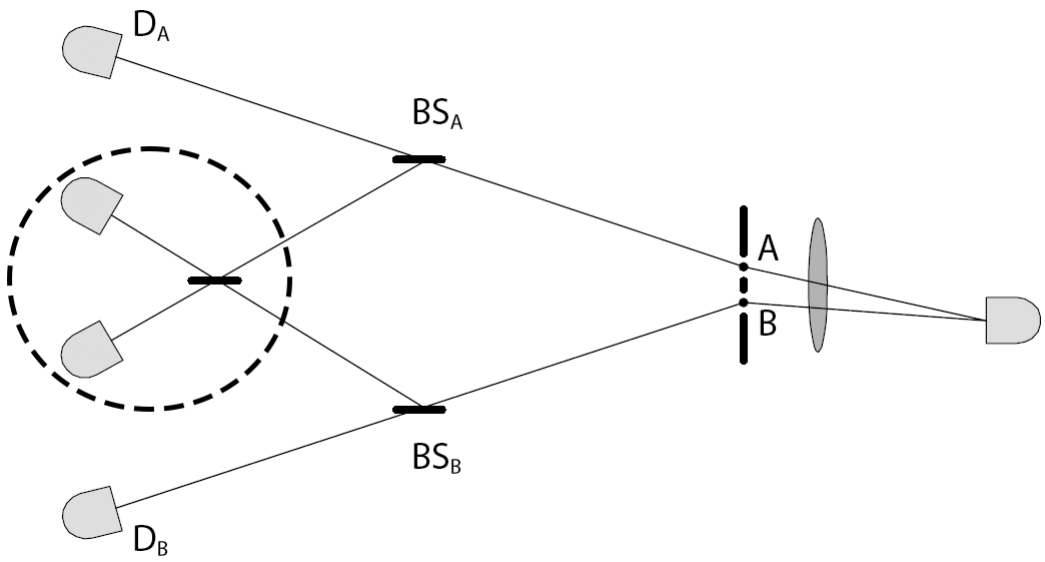
Even without including quantum gravitational effects directly, we can explore ways to bend the notion of causality within quantum mechanics itself. In [44] it is shown that general correlations between two systems do not necessarily possess a causal order. This is done by exploring the class of all possible correlations, restricted only by the axioms of local quantum mechanics. Importantly, this is done without introducing a global structure of spacetime, which in quantum mechanical systems is often imposed via causality arguments - e.g. that a system cannot both receive and send information to another at the same time -. The proof that some correlations cannot be assigned any causal order can be achieved by considering the following task: two separate systems, A and B, each produce a random bit, then one of them, chosen at random, has to guess the other's bit. Suppose we assume some causal order

of these events, for example that system A produces its bit before B and they are causally connected, so that A can communicate its bit to B but not vice versa. It is easy to calculate the probability of success in this case: if A has to guess then this has to be random - there is no way to get information on B's run since it is in the causal future -; whereas if B has to guess they can receive information on which bit system A has produced in its run, so they will always be able to guess correctly. The probability of success therefore comes out to 3/4. It turns out that this is the maximum possible probability for this task under any physically possible causal relation. Indeed, the other available setups are the equivalent reverse one - still time-like separated but with A and B exchanged -, which of course does not change the probability; and space-like separation, which reduces the rate of success to 1/2 since the two guesses both have to be random. Next, the authors provide an example of a process, inside the more general class of correlations described above, for which the probability of success comes out to $(2 + \sqrt{2})/4 > 3/4$. This means that the process under consideration cannot possess a causal order, something which would have profound consequences on the nature of time. Of course, these types of processes might still only be possible under exotic conditions involving quantum gravitational effects, since they are obtained by relaxing the assumption of a "regular" background spacetime, with its global causal structure.

Remaining now within the predictions of standard quantum theory, we can still find some exclusively quantum effects which have brought a substantial number of discussions on the nature of causality. The main example of these is the phenomenon of entanglement. As discussed in the previous section, entanglement has urged many physicists to abandon the principle of locality. More specifically, the famous Bell inequalities [13], along with the actual experiments that would confirm their violation in quantum physics, had profound consequences on what a valid physical theory can look like. Hidden-variable theories which exhibit locality - along with a few other reasonable assumptions - were shown to be inconsistent with the results. However, this is by no means the only valid pursuit. For example,

in [14] Bell himself discusses - and is decidedly against - the possibility that the experimental setups, even possibly the experimenters' decisions, are correlated with the system under observation, and that this "conspiracy" causes the system's behaviour to be influenced by more than just local interactions. These hidden variables would have to be non-local, and possibly even retroactive, in order to consistently explain all phenomena. For a dramatic illustration of this, we now analyse the quantum eraser experiment.

The setup for the so-called delayed "choice" quantum eraser, first proposed in [53] and as performed by Kim et al. [39], is shown schematically in the figure below.



Delayed "Choice" Quantum Eraser Experiment. Either atom A or atom B emits a pair of photons, one to the left and one to the right. The latter is detected on a screen, the former sent to a beam splitter, either BS_A or BS_B depending on the atom it was emitted from. Here the photon is sent either to a detector, D_A or D_B depending on the splitter, or to an "eraser" (inside the dashed outline). From [39].

A pair of atoms, A and B, are stimulated so as to emit a pair of entangled photons. Knowledge of which atom emits the pair is akin to "which-path" information in this sense: one of the photons can be detected on a screen and, if no which-path information is obtained,

by detecting many of these events we can observe an interference pattern. The two atoms therefore play the same role as the slits in the standard double-slit setup. What differentiates this setup is the delayed choice, made by a pair of beam splitters BS_A and BS_B which will receive the other photon in the entangled pair from A and B respectively. These randomly send the photon either directly to a detector (D_A or D_B), which enables us to determine which-path information, or to an eraser. The eraser is composed of a beam splitter which takes photons from both BS_A and BS_B and sends them to one of two detectors at the other end. When one of these two detectors clicks, because of the presence of the splitter it is impossible to determine which atom the photon was emitted from, therefore we say that the which-path information has been erased.

The goal of this experiment is to determine if one can still observe interference, even when the choice of whether or not to erase the which-path information is made after the detection itself. Standard quantum mechanics predicts that this is indeed possible, as shown in the original proposal [53]. We therefore need to make sure that the choice is made after each screen detection. To achieve this, the beam splitters BS_A and BS_B, which are what effectively chooses between erasing and recording the information, have to be at a much larger distance from the atoms compared to the screen. What one finds, when analysing the screen detections corresponding to the photons which were entangled to the erased ones, is that indeed we have interference. Conversely, when looking at photons entangled with ones that arrived at D_A or D_B - which determine the which-path information -, no interference is present. Note that the presence of two detectors in the eraser is important, as we can only see interference by observing the photons that arrived in each one separately. The two patterns together instead precisely overlap with each other, so that no interference can be observed.

These results, depending on one's view of quantum mechanics, can seem quite remarkable.

The standard double-slit experiment has striking implications regarding locality and complementarity, but this experiment has the added peculiarity of having seemingly retrocausal effects. The knowledge of whether or not the entangled partner will be deflected towards an eraser appears to be already contained in the photon when it is detected on the screen. Of course, this setup does not allow signalling backwards in time. In the same vein as the superluminal effects related to entanglement or tunnelling described before, although quantum mechanics displays non-local correlations between systems, no acausal effect is needed to explain them. In this case, as mentioned, no interference pattern can be observed when all detections are considered indiscriminately. Therefore, as for entanglement, only a causal connection between the two systems (the screen and the far away detectors) allows one to obtain information. The seemingly retroactive influence by the photons detected on the screen can be avoided via the following arguments, again all within the standard interpretation. When the photons are detected on screen their distribution already contains the potential interference effect, independent of whether or not it will be erased later. Crucially, this looks exactly like no interference, as discussed. It is, however, not entirely the same as a standard which-path experiment, as some kind of interference does in fact take place before detection, but it does so in a way which is not readily observable. In other words, the photons being detected on the screen are not the same as those of a double slit setup, but they are also unlike those of a which-path experiment. We have to allow for a more complicated behaviour of the wave-function than just "either particle or wave". Only the entangled partners have this information, and depending on the way we measure their positions it can be destroyed or detected. In the first case, the information is irretrievably lost. In the second, by decoherence the information leaks into the environment and we can therefore observe an interference pattern from the noise by selecting the right photons. From this prospective, this experiment is similar to standard entanglement experiments, in that non-local effects are all that is needed for a proper explanation. Some interpretations such as many worlds are able to avoid both non-locality and the apparently retroactive influence.

5.4 The Problem of Time in Quantum Gravity

Investigations of quantum gravity have caused a resurfacing of many open issues in the foundations of quantum mechanics, but they have also brought their share of problems to solve. The main problem concerning time that appears in many theories of quantum gravity stems from the differences in the treatment of time in relativity and quantum mechanics. When trying to canonically quantize GR, starting from the classical Hamiltonian constraint $\mathcal{H} = 0$ and promoting it to a quantum constraint, one is led to the Wheeler-DeWitt Equation [30]:

$$\hat{\mathcal{H}}\Psi = 0. \tag{5.7}$$

Note the striking absence of time in this equation, quite unlike a standard Schrödinger type equation. This is where the problem of time stems from: in a theory where the wave-function itself only depends on the configurations of the gravitational and matter fields, and not on time, how does the latter arise? Importantly, there are reasons to believe this is not related to the renormalization problem these theories often have. Simpler finite-dimensional toy models without infinities still exhibit these difficulties [36], which constitute a conceptual problem of interpretation rather than one of results of calculations. The possible approaches generally fall into two main categories [11]:

1) Time is fundamental and therefore should be included in the theory either at the classical level or emerge in some way after quantization.

2) The universe as a whole is timeless, and physics can, at the right level, be carried out without time.

Examples of theories of the first type are internal time theories, where for instance matter or a cosmological constant play the role of a time reference. Theories of the second type include the conditional probabilities interpretation and the various records schemes, which

try to derive information about dynamics from correlations between so-called "records" at a single instant. Much progress has been and is still being carried out on these and many other approaches to solve the problem of time, which is still one of the central conceptual problems in the fields of quantum gravity and quantum cosmology.

6 Time and Interpretations of Quantum Mechanics

Quantum theory is, to this day, very problematic when it comes to its implications on the nature of reality. Does collapse fundamentally occur and what is the precise mechanism behind it? Can we consistently describe the whole universe as a quantum system, and how can we describe observers and measurements within this description? Are some objects, such as classical observers, indescribable in quantum theory and necessarily separate from it? Does a more fundamental theory exist, which will resolve all these issues and go back to the paradigm of classical theories? The various interpretations born over the decades since the theory's conception attempt to answer some or all of these questions. The precise nature of time depends, to some degree, on the answers.

6.1 Copenhagen, Objective-Collapse

Although there are many variations of what we can refer to as the standard interpretation of quantum mechanics, a few main features are always present. The important point as regards the nature of time in this interpretation is that measurement constitutes a sudden and irreversible process. Experiments on quantum systems are described from the point of view of classical observers, which is why there has to be a definite line somewhere between the quantum and classical regimes (again, the details of this can vary, but it is a general feature of the interpretation that this divide exists). After interaction with a classical system, a quantum system's behaviour changes radically: the usual unitary evolution, which takes place when the system is isolated, is replaced by a sudden and non-linear collapse of the wave-function to whichever state is compatible with the measured value. This phenomenon gives time a clear direction, since the wave-function instantly collapses (as opposed to suddenly spreading out) as time increases.

In objective-collapse models, which can be regarded as separate theories rather than interpretations, the wave function collapses objectively and independently of observations, with the precise cause and mechanism depending on the specific model. In GRW theory [28], as an example, collapse is modelled stochastically. It happens spontaneously and proportionally to the amplitude squared of the wave-function, in accordance with the Born rule. Apart from these events, the particles otherwise follow the Schrödinger equation at all times. Each particle separately has a chance to collapse, which in a large system causes the centre of mass to collapse. This so-called amplification mechanism would explain why large systems do not exhibit quantum behaviour. Because collapse is Poissonian in time, we have to introduce a new fundamental constant λ with dimensions of frequency, corresponding to the rate of collapse, so that there is a probability λdt of collapse occurring in a time dt. Most of these theories are not that different from the standard interpretation when it comes to the role of the external parameter of time, except here it plays another fundamental function, other than in the usual wave-function dynamics, as the parameter in the dynamics that describe collapse.

6.2 Hidden Variables

The many hidden-variable theories all postulate additional variables that determine the behaviour of the system in a stronger way than standard quantum mechanics allows. Our current description in terms of wave-functions and the Born rule is considered incomplete, but the extra hidden variables are often inaccessible. The most famous hidden-variable theory is the de Broglie-Bohm or pilot wave theory [15]. What is hidden in this model are the actual positions and velocities of the point-like particles, which take specific real values at every moment in time. This is almost completely equivalent to the description of a classical universe, except the motion of particle "i" in an N-particle system with wave-function $\Psi(\mathbf{x}, t)$

is now given by

$$\frac{d\mathbf{x}_i}{dt} = \frac{\mathbf{J}_i}{|\Psi|^2}, \tag{6.1}$$

where $\mathbf{J}_i$ is the usual probability current associated with particle i, and the wave-function Ψ solves the many-particle Schrödinger equation. This formulation actually makes the predictions of this description completely equivalent to those of standard quantum mechanics. However, here the particles are interpreted to have a real location and velocity at each moment in time. In light of Bell's theorem discussed earlier, such a theory must somehow include non-local effects. These are indeed already present in (6.1), as the velocity of a specific particle depends on the entire wave-function, which contains information about all the particles in the system at a given time. Although the motion gives this theory all the required quantum effects, it is in all other interpretational aspects, including that of time, entirely analogous to a classical theory.

6.3 Many Worlds

Another completely deterministic reading of quantum mechanics is Everett's many-worlds interpretation [35]. In it, the wave-function of the universe evolves according to the Schrödinger equation at all times. All possible outcomes of a measurement are always realised, and collapse is seen as only an effective description, due to decoherence and the fact that the observer is part of the universal wave-function. Tough very much in line with the perspective of quantum cosmology, this approach becomes problematic as the notions of measurement, collapse and even the Born rule are not assumed, and have to be derived by considering interactions between sub-systems inside the universal wave-function. For example, many different approaches to deriving the Born rule have been proposed over the years [60, 24, 32].

At any rate, the phenomenon of collapse which breaks the time symmetry of standard quan-

tum mechanics is entirely absent here. The splitting of the universal wave-function at each quantum event still appears to give time a clear direction. But if we follow this splitting to its cause, it points to the same source as the "thermal time" hypothesis: the relatively low entropy - and low entanglement - state of the early universe. Thus, time is completely symmetric in this theory, and only acquires directionality depending on the characteristics of a state.

6.4 The Two-State Vector Formalism

Many other theories attempt to preserve the internal time symmetry of quantum theory in different ways. The two-state vector formalism [3] is one such approach, completely equivalent to standard quantum theory, but operationally and conceptually very different. In it, the complete description of a quantum state is not given by just the usual state but also by an additional state evolving backwards in time. These two states are usually described in terms of pre- and post-selection. Just as pre-selection by measurement of eigenvalue λ - with associated state $|\lambda\rangle$ - creates a forwards evolving state in the usual way

$$|\Psi(t)\rangle = e^{-i\hat{H}t}|\lambda\rangle, \tag{6.2}$$

post-selection similarly creates a backwards evolving state

$$\langle\Phi(t)| = \langle\lambda|e^{+i\hat{H}t}. \tag{6.3}$$

The complete state when pre- and post-selecting with an arbitrary pair of measurements at times t_1 and t_2 respectively, is given for all times t with $t_1 < t < t_2$, by both of these:

$$\langle\Phi(t)| \quad |\Psi(t)\rangle. \tag{6.4}$$

The above is what is called a *two-state vector*. In practice, using this we can calculate the probability of obtaining the measurement outcome λ, when the measurement occurs between t_1 and t_2, via the following rule:

$$p(\hat{P}_\lambda) = \frac{|\langle\Phi|\hat{P}_\lambda|\Psi\rangle|^2}{\sum_j |\langle\Phi|\hat{P}_j|\Psi\rangle|^2}, \tag{6.5}$$

where $\hat{P}_j$ are all the projections associated with the observable in question and $\hat{P}_\lambda$ is the one associated with the outcome λ.

It might appear that the previous formula gives a different prediction from that of standard quantum mechanics. However, the whole procedure does not simply consist of pre-selection and measurement, but also includes post-selection. Thus, the probability being calculated is that of obtaining outcome λ given both pre- and post-selection of the state. In practice - as expected for post-selection -, this has to be achieved by the use of ensembles, by performing the required measurements and then considering only the ones where the right state was post-selected. The advantage of this formalism is being able to concisely describe a pre- and post-selected state and the measurements performed on it. Some interesting phenomena have been discovered using it, such as weak measurements [2] and the three-boxes paradox [6]. The latter is a good illustration of the advantages and difficulties of this approach, and can be illustrated as follows.

Consider a system with zero Hamiltonian and a two-state vector given by

$$\langle\Phi| \quad |\Psi\rangle = \frac{1}{3}(\langle 1| + \langle 2| - \langle 3|) \ (|1\rangle + |2\rangle + |3\rangle), \tag{6.6}$$

where the states $|i\rangle$, representing the three boxes, are orthogonal and normalized. The projection operators corresponding to the measurement of box 1 are:

$$\hat{P}_1^1 = |1\rangle\langle 1|, \qquad \hat{P}_0^1 = |2\rangle\langle 2| + |3\rangle\langle 3|. \tag{6.7}$$

They correspond to the eigenstates of finding the particle in box 1 or not, respectively, when opening it. Applying (6.5) for the probability of finding the particle in box 1 when in the state (6.6), we obtain $p(\hat{P}_1^1) = 1$, so we are certain to find the particle when we open box 1. It might be tempting to interpret this as the particle *being* in box 1 for all times between pre- and post-selection. However, if we define the operators $\hat{P}_1^2$ and $\hat{P}_0^2$ for the opening of box 2, similar to (6.7), we also find that $p(\hat{P}_1^2) = 1$, which immediately makes this interpretation inconsistent. These variables with guaranteed outcomes, which have been termed "elements of reality", are only real in the classical sense when the corresponding measurement is performed. As for many concepts in quantum mechanics, no ontological meaning should be assigned to them independently of measurements, so as to avoid contradictions and paradoxes such as the one we have just described.

Although this approach can be turned into a separate interpretation of quantum mechanics [4], what we have discussed so far is entirely in agreement with the standard theory, and can be rephrased in those terms. Using the language of ensembles, what the calculations above tell us is that in the sequence of pre-selecting state $|\Psi\rangle$, opening box 1, and then post-selecting state $|\Phi\rangle$, we are guaranteed to find the particle when opening box 1, but only in those elements of the ensemble which we happen to correctly post-select to $|\Phi\rangle$. What constitutes the paradox is that this exact same prediction also happens to be valid for box 2.

This difference between pre- and post-selection might seem to indicate an asymmetry in

time, but in fact we can trace it back to the memory's arrow of time. We can always successfully pre-select a state and only perform the measurements on the states we prefer, whereas we always need to account for the possible failure of post-selecting a state. This is only the case because the measurement happens before post-selection, and the concept of measurement itself is still time-asymmetric: we do not know the result before a measurement, and we know it after. As discussed, we can only solve this post-selection problem by using an ensemble and then only considering the systems where the correct post-selection has been made. However, as the authors in [7] clarify, the concept of *measurement results* is actually time-symmetric. This is because it quantifies a time-symmetric interaction between two systems, or, more practically the shift of a pointer, which as an absolute quantity is independent of the direction of time.

7 Conclusion

In this dissertation, we have analysed some problems regarding the nature of time in quantum mechanics. Firstly, we have reviewed the trend of revising our understanding of time throughout history, a trend which certainly shows no signs of slowing down.

Then, we have described the ways in which measurements influence a system's evolution, which is of central importance in time measurements. The phenomenon of decoherence tells us exactly which kinds of states will be suppressed and which will be reinforced when interacting with the environment. The theory of weak measurements emphasizes the link between information gained and disturbance caused on a system. The quantum Zeno effect, whose precise limits and applications are still being explored today, provides the most dramatic example of how collapse changes the future behaviour of wave-functions. These concepts can be observed in the phenomenon of tunnelling, which exemplifies how seemingly unambiguous quantities such as the time of traversal or the tunnelling time are in fact difficult to describe in the quantum framework. A precise description of these quantities has to involve specific measurement schemes, with different schemes often yielding different results. It is therefore no surprise that investigations of these matters have shed some light into the measurement problem and the relationship between operators and measurements, and they will hopefully continue to do so in the future.

We have seen how the time operator problem was clarified by Pauli's theorem, and how more recent discoveries such as POVMs and extensions to the CCR can provide some solutions. Further alternative approaches like the path integral formulation can serve as different ways of obtaining time distributions. This approach has also helped us identify similarities between measuring the dwell time and measuring which-path information, in that a similar "selection" of specific paths occurs in both situations. All these approaches to the time operator problem show that it is not unequivocal and absolute, but rather both mathematically

and physically sophisticated and system-dependent. Exploring its ramifications can mean discovering more about operators and observables in general.

Next, we have analysed the notion of external time, which is central to this theory, and the fact that the existence of a device that can perfectly track such a quantity is actually forbidden by the theory itself. We have explored the phenomenon of entanglement, how a similar notion can be applied to correlations in time, and the similarities and differences between the two. We have examined the role of causality in quantum mechanics and beyond. Despite some seemingly counterintuitive results, the consensus is that this principle has so far never been violated in this theory. On the other hand, some hints from possible theories of quantum gravity are casting doubts on its universality. This interface between the foundations of physics and gravity certainly has the potential to provide more clues on what a complete theory of quantum gravity might look like. Related to this is the problem of time, which represents an important difficulty of quantum cosmology and for which many approaches are still being devised and discussed.

Lastly, we have reviewed the properties of time in some interpretations of quantum mechanics. Since the debate on these is far from settled, we cannot know how time might look like in future theories, as starkly different "kinds" of time are still allowed by the various interpretations: from the entirely deterministic universe of hidden-variable theories, to the splitting universes encompassed by the many worlds interpretation, to the completely time-symmetric model of the two-state vector formalism.

All these ideas point to a concept of time that is ever evolving, but about which we are discovering more and more, and which remains of central importance in all our descriptions of the physical world.

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